MMAT5390: Mathematical Image Processing

Assignment 1

Due: 23:59 Wednesday, February 5, 2025

Please give reasons in your solutions unless otherwise specified.

1. Suppose $\mathcal{O}$ is a separable image transformation such that its point spread function is given by:
$$h^{\alpha,\beta}(x,y) = A(\alpha,x)B(y,\beta),$$
where $A \in M_{n\times n}(\mathbb{R})$, $B \in M_{n\times n}(\mathbb{R})$ and $n \in \mathbb{N}$.
Show that the transformation matrix H is given by:
$$H = B^T \otimes A.$$
Here, $\otimes$ means the Kronecker product:
$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{pmatrix}.$$

2. Suppose $H \in M_{n^2\times n^2}(\mathbb{R})$ is the transformation matrix of a shift-invariant linear image transformation. Further suppose that the image is periodically extended. Show that H is block-circulant.
i.e.
$$H = \begin{pmatrix} A_1 & A_n & \cdots & A_2 \\ A_2 & A_1 & \cdots & A_3 \\ \vdots & \vdots & \ddots & \vdots \\ A_n & A_{n-1} & \cdots & A_1 \end{pmatrix},$$
where $A_i \in M_{n\times n}(\mathbb{R}) \quad \forall i = 1, 2, \cdots, n$.

3. (a) Suppose $\mathcal{O}$ is a separable image transformation such that
$$\begin{aligned}\mathcal{O}(f) &= AfB \\ &= \begin{pmatrix} 1 & 0 & 3 \\ 0 & 5 & 0 \\ 7 & 0 & 9 \end{pmatrix} f \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix}\end{aligned}$$
Find the transformation matrix H of this image transformation.

   (b) Given the transformation matrices H below. Determine if the following image transformations $\mathcal{O}$ are separable and/or shift-invariant.

   i. $H = \begin{pmatrix} \pi & \pi^2 & \pi^4 & \pi^3 \\ \pi^3 & \pi^4 & \pi^2 & \pi \\ \pi^4 & \pi^3 & \pi & \pi^2 \\ \pi^2 & \pi & \pi^3 & \pi^4 \end{pmatrix}$

   ii. $H = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 10 & 0 & 10 & 0 & 0 & 0 & 4 & 0 & 4 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 & 0 & 4 & 0 \\ 2 & 0 & 2 & 6 & 0 & 6 & 8 & 0 & 8 \\ 0 & 1 & 0 & 0 & 6 & 0 & 0 & 4 & 0 \end{pmatrix}$

4. Consider a linear image transformation $O : M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ defined by:

$$O(f)(\alpha,\beta) = \frac{1}{4}\left[-5f(\alpha,\beta) + 2f(\alpha+1,\beta) + 3f(\alpha-1,\beta) + 4f(\alpha,\beta+1) + f(\alpha,\beta-1)\right],$$

for all $1 \le \alpha, \beta \le N$ and $f \in M_{N\times N}(\mathbb{R})$. Show that the transformation $O(f)$ can be represented as a convolution of the image f with a suitable kernel $k \in M_{N\times N}(\mathbb{R})$, i.e. $O(f) = k * f$. Identify the kernel k explicitly.

5. Let $f = \text{circ}((1,2,3)^T) = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$.

   (a) Compute the Singular Value Decomposition (SVD) of f.

   (b) Express f as a linear combination of its elementary images.

   (c) Consider the matrix $g = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$, which is a permutation of f. Discuss the relationship between the SVDs of f and g.